Outline of the Solution

We proceed in six main steps:

Step 1: Construct A and form $B = A - A^T$. Record the distribution of B_{ij}.

Step 2: Recall how $\det(B)$ expands and note why any "unpaired" factor has zero mean.

Step 3: Invoke the Pfaffian–squared identity for skew–symmetric matrices.

Step 4: Count the number of perfect matchings (the surviving square–terms).

Step 5: Compute the contribution of each square term via $\mathbb{E}[B_{ij}^2]$.

Step 6: Combine counts and contributions to reach a closed-form formula.

Step 1: From A to $B = A - A^T$

- A is $2n \times 2n$, each $A_{ij} \in \{0, 1\}$ by a fair coin flip (prob. $\frac{1}{2}$).
- $(A^T)_{ij} = A_{ji}$.
- Define
$$B = A - A^T, \quad B_{ij} = A_{ij} - A_{ji}.$$
- On the diagonal $B_{ii} = 0$. Off–diagonal $B_{ij} \in \{+1, 0, -1\}$.

Distribution of each B_{ij}

Each pair (A_{ij}, A_{ji}) has four equally likely outcomes:

A_{ij}	A_{ji}	$B_{ij} = A_{ij} - A_{ji}$	Probability
1	0	$+1$	$\frac{1}{4}$
0	1	-1	$\frac{1}{4}$
0	0	0	$\frac{1}{4}$
1	1	0	$\frac{1}{4}$

Therefore:

$$\Pr(B_{ij} = +1) = \tfrac{1}{4}, \quad \Pr(B_{ij} = -1) = \tfrac{1}{4}, \quad \Pr(B_{ij} = 0) = \tfrac{1}{2}.$$

Independence gives

$$\mathbb{E}[B_{ij}] = 0, \qquad \mathbb{E}[B_{ij}^2] = \tfrac{1}{2}.$$

Step 2: Determinant Expansion

For a square matrix M, $\det(M)$ expands as a sum of signed products, each choosing one entry from every row and column. In our case, since $B_{ii} = 0$, every product uses exactly n off–diagonal entries B_{ij}.

Key fact: A random variable X with $\mathbb{E}[X] = 0$ forces

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X]\,\mathbb{E}[Y] = 0$$

for any independent Y. Hence any product containing a factor B_{ij} only once has expectation zero.

Why $\mathbb{E}[B_{ij}] = 0$

Recall

$$B_{ij} = A_{ij} - A_{ji},$$

where A_{ij}, A_{ji} are independent fair flips in $\{0, 1\}$. The four equally likely outcomes give

(A_{ij}, A_{ji})	B_{ij}	Pr
$(1, 0)$	$+1$	$\frac{1}{4}$
$(0, 1)$	-1	$\frac{1}{4}$
$(0, 0)$	0	$\frac{1}{4}$
$(1, 1)$	0	$\frac{1}{4}$

Thus

$$\mathbb{E}[B_{ij}] = (+1) \cdot \tfrac{1}{4} + (-1) \cdot \tfrac{1}{4} + 0 \cdot \tfrac{1}{2} = 0.$$

Example of a vanishing term

Let $n = 2$ so B is 4×4. Consider the product

$$B_{12}\; B_{23}\; B_{34}\; B_{41}.$$

Each factor has mean zero and they are independent, therefore

$$\mathbb{E}[B_{12}B_{23}B_{34}B_{41}] = \mathbb{E}[B_{12}]\,\mathbb{E}[B_{23}]\,\mathbb{E}[B_{34}]\,\mathbb{E}[B_{41}] = 0.$$

Only square–terms survive: The only terms with nonzero expectation are those in which every B_{ij} appears twice, e.g.

$$\big(B_{i_1 j_1} B_{i_2 j_2} \cdots B_{i_n j_n}\big)^2,$$

since then each factor becomes B_{ij}^2 and $\mathbb{E}[B_{ij}^2] = \frac{1}{2} \neq 0$.

Step 3: Pfaffian–Squared Identity

A matrix B is called *skew–symmetric* if $B^T = -B$. In that case a beautiful fact (and extremely useful for us!) holds:

$$\det(B) = \big[\mathrm{Pf}(B)\big]^2.$$

Here's what that means:

3.1 Defining the Pfaffian

1. Label the rows and columns $1, 2, \ldots, 2n$.
2. A *perfect matching* π is a way to pair up these $2n$ indices into n unordered pairs $\{(i_1, j_1), \ldots, (i_n, j_n)\}$.
3. For each matching π, form the product

$$\prod_{(i,j)\in\pi} B_{ij}.$$

4. Assign a sign $\mathrm{sgn}(\pi) \in \{+1, -1\}$ to π, which you can think of as $+1$ if an even number of swaps of indices is needed to reorder $(i_1, j_1, i_2, j_2, \ldots)$ into $(1, 2, 3, 4, \ldots)$, and -1 otherwise.
5. Then

$$\mathrm{Pf}(B) = \sum_{\pi \text{ perfect matching}} \mathrm{sgn}(\pi) \prod_{(i,j)\in\pi} B_{ij}.$$

3.2 Squaring the Pfaffian

When we square $\mathrm{Pf}(B)$, we get a double-sum over two matchings π, σ:

$$\det(B) = \big[\mathrm{Pf}(B)\big]^2 = \sum_{\pi,\sigma} \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma) \Big(\prod_{(i,j)\in\pi} B_{ij} \Big)\Big(\prod_{(k,\ell)\in\sigma} B_{k\ell} \Big).$$

- If $\pi \neq \sigma$, at least one B_{ij} appears only once in the combined product. Since $\mathbb{E}[B_{ij}] = 0$, that term vanishes when we take the expectation.
- If $\pi = \sigma$, we get

$$\Big(\prod_{(i,j)\in\pi} B_{ij} \Big)^2 = \prod_{(i,j)\in\pi} B_{ij}^2,$$

and each B_{ij}^2 has nonzero mean $\frac{1}{2}$.

Step 4: Counting Perfect Matchings

We need to know how many ways there are to pair the $2n$ indices $\{1, 2, \ldots, 2n\}$ into n unordered pairs. Two equivalent arguments:

Argument A: Pairing Up

1. Start with label 1. It can pair with any of the other $2n - 1$ labels. (choices: $2n - 1$)
2. Remove that pair. Now the smallest remaining label (say 2) can pair with any of the other $2n - 3$ labels. (choices: $2n - 3$)
3. Continue: at each step you pick a partner for the smallest unpaired label, with $2n - 5$, $2n - 7$, ..., down to 1 choice.
4. Multiply the choices:

$$(2n-1) \times (2n-3) \times (2n-5) \times \cdots \times 3 \times 1 = (2n-1)!!.$$

Argument B: Divide out overcounts

$$\#\{\text{pairings}\} = \frac{\text{all orderings of } 2n \text{ items}}{\substack{\text{order inside each pair} \\ \text{doesn't matter}} \times \substack{\text{order of the pairs} \\ \text{doesn't matter}}} = \frac{(2n)!}{2^n\, n!} = (2n-1)!!.$$

Here

- $(2n)!$: ways to line up all $2n$ labels.
- Divide by 2^n: within each of the n pairs the two elements can swap.
- Divide by $n!$: the order of the n pairs themselves does not matter.

Hence the number of "square–terms" in our determinant expansion is

$$(2n-1)!! \;=\; 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

Step 5: Contribution of Each Square–Term

Each surviving term comes from squaring one matching's product:

$$\big(B_{i_1 j_1}\, B_{i_2 j_2}\, \cdots\, B_{i_n j_n}\big)^2 = B_{i_1 j_1}^2\, B_{i_2 j_2}^2 \cdots B_{i_n j_n}^2.$$

Because the B_{ij} are independent,

$$\mathbb{E}\big[B_{i_1 j_1}^2 \cdots B_{i_n j_n}^2\big] = \prod_{r=1}^{n} \mathbb{E}[\, B_{i_r j_r}^2\,].$$

But each B_{ij}^2 is either 1 (when $B_{ij} = \pm 1$) or 0, and we saw

$$\mathbb{E}[B_{ij}^2] = 1 \cdot \Pr(B_{ij} = \pm 1) + 0 \cdot \Pr(B_{ij} = 0) = \frac{1}{2}.$$

Therefore every square–term contributes

$$\left(\tfrac{1}{2}\right)^n$$

to the overall expectation.

Step 6: Final Result

Putting together

$$\underbrace{(2n-1)!!}_{\text{number of pairings}} \qquad \text{times} \qquad \underbrace{\left(\tfrac{1}{2}\right)^n}_{\text{contribution each}} \quad \longrightarrow \; \mathbb{E}\big[\det(A - A^T)\big] = (2n-1)!!\, 2^{-n}.$$

Alternatively,

$$(2n-1)!! = \frac{(2n)!}{2^n\, n!} \quad \Longrightarrow \quad \mathbb{E}[\det(A - A^T)] = \frac{(2n)!}{4^n\, n!}.$$

Check for small n

$$n = 1 : \; \tfrac{1}{2}, \qquad n = 2 : \; \tfrac{3}{4}, \qquad n = 3 : \; \tfrac{15}{8}, \ldots$$

This completes the solution.